

# HERMITE-HADAMARD-TYPE INEQUALITIES FOR $(g, \varphi_h)$ -CONVEX DOMINATED FUNCTIONS

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**ABSTRACT.** In this paper, we introduce the notion of  $(g, \varphi_h)$ -convex dominated function and present some properties of them. Finally, we present a version of Hermite-Hadamard-type inequalities for  $(g, \varphi_h)$ -convex dominated functions. Our results generalize the Hermite-Hadamard-type inequalities in [2], [4] and [6].

## 1. INTRODUCTION

The inequality

$$(1.1) \quad f\left(\frac{a+b}{2}\right) \leq \frac{1}{b-a} \int_a^b f(x) dx \leq \frac{f(a) + f(b)}{2}$$

which holds for all convex functions  $f : [a, b] \rightarrow \mathbb{R}$ , is known in the literature as Hermite-Hadamard's inequality.

In [1], Dragomir and Ionescu introduced the following class of functions.

**Definition 1.** Let  $g : I \rightarrow \mathbb{R}$  be a convex function on the interval  $I$ . The function  $f : I \rightarrow \mathbb{R}$  is called  $g$ -convex dominated on  $I$  if the following condition is satisfied:

$$|\lambda f(x) + (1-\lambda)f(y) - f(\lambda x + (1-\lambda)y)|$$

$$\leq \lambda g(x) + (1-\lambda)g(y) - g(\lambda x + (1-\lambda)y)$$

for all  $x, y \in I$  and  $\lambda \in [0, 1]$ .

In [2], Dragomir *et al.* proved the following theorem for  $g$ -convex dominated functions related to (1.1).

Let  $g : I \rightarrow \mathbb{R}$  be a convex function and  $f : I \rightarrow \mathbb{R}$  be a  $g$ -convex dominated mapping. Then, for all  $a, b \in I$  with  $a < b$ ,

$$\left| f\left(\frac{a+b}{2}\right) - \frac{1}{b-a} \int_a^b f(x) dx \right| \leq \frac{1}{b-a} \int_a^b g(x) dx - g\left(\frac{a+b}{2}\right)$$

and

$$\left| \frac{f(a) + f(b)}{2} - \frac{1}{b-a} \int_a^b f(x) dx \right| \leq \frac{g(a) + g(b)}{2} - \frac{1}{b-a} \int_a^b g(x) dx.$$

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In [1] and [2], the authors connect together some disparate threads through a Hermite-Hadamard motif. The first of these threads is the unifying concept of a  $g$ -convex-dominated function. In [3], Hwang *et al.* established some inequalities of Fejér type for  $g$ -convex-dominated functions. Finally, in [4], [5] and [6] authors introduced several new different kinds of convex -dominated functions and then gave Hermite-Hadamard-type inequalities for this classes of functions.

In [7], S. Varošanec introduced the following class of functions.

$I$  and  $J$  are intervals in  $\mathbb{R}$ ,  $(0, 1) \subseteq J$  and functions  $h$  and  $f$  are real non-negative functions defined on  $J$  and  $I$ , respectively.

**Definition 2.** Let  $h : J \rightarrow \mathbb{R}$  be a non-negative function,  $h \not\equiv 0$ . We say that  $f : I \rightarrow \mathbb{R}$  is an  $h$ -convex function, or that  $f$  belongs to the class  $SX(h, I)$ , if  $f$  is non-negative and for all  $x, y \in I$ ,  $\alpha \in (0, 1]$ , we have

$$(1.2) \quad f(\alpha x + (1 - \alpha)y) \leq h(\alpha)f(x) + h(1 - \alpha)f(y).$$

If the inequality (1.2) is reversed, then  $f$  is said to be  $h$ -concave, i.e.  $f \in SV(h, I)$ .

Youness have defined the  $\varphi$ -convex functions in [9]. A function  $\varphi : [a, b] \rightarrow [c, d]$  where  $[a, b] \subset \mathbb{R}$ :

**Definition 3.** A function  $f : [a, b] \rightarrow \mathbb{R}$  is said to be  $\varphi$ -convex on  $[a, b]$  if for every two points  $x \in [a, b]$ ,  $y \in [a, b]$  and  $t \in [0, 1]$  the following inequality holds:

$$f(t\varphi(x) + (1 - t)\varphi(y)) \leq tf(\varphi(x)) + (1 - t)f(\varphi(y)).$$

In [8], Sarıkaya defined a new kind of  $\varphi$ -convexity using  $h$ -convexity as following:

**Definition 4.** Let  $I$  be an interval in  $\mathbb{R}$  and  $h : (0, 1) \rightarrow (0, \infty)$  be a given function. We say that a function  $f : I \rightarrow [0, \infty)$  is  $\varphi_h$ -convex if

$$(1.3) \quad f(t\varphi(x) + (1 - t)\varphi(y)) \leq h(t)f(\varphi(x)) + h(1 - t)f(\varphi(y))$$

for all  $x, y \in I$  and  $t \in (0, 1)$ .

If inequality (1.3) is reversed, then  $f$  is said to be  $\varphi_h$ -concave. In particular, if  $f$  satisfies (1.3) with  $h(t) = t$ ,  $h(t) = t^s$  ( $s \in (0, 1)$ ),  $h(t) = \frac{1}{t}$ , and  $h(t) = 1$ , then  $f$  is said to be  $\varphi$ -convex,  $\varphi_s$ -convex,  $\varphi$ -Godunova-Levin function and  $\varphi$ - $P$ -function, respectively.

In the following sections our main results are given: We introduce the notion of  $(g, \varphi_h)$ -convex dominated function and present some properties of them. Finally, we present a version of Hermite-Hadamard-type inequalities for  $(g, \varphi_h)$ -convex dominated functions. Our results generalize the Hermite-Hadamard-type inequalities in [2], [4] and [6].

## 2. $(g, \varphi_h)$ -CONVEX DOMINATED FUNCTIONS

**Definition 5.** Let  $h : (0, 1) \rightarrow (0, \infty)$  be a given function,  $g : I \rightarrow [0, \infty)$  be a given  $\varphi_h$ -convex function. The real function  $f : I \rightarrow [0, \infty)$  is called  $(g, \varphi_h)$ -convex dominated on  $I$  if the following condition is satisfied

$$(2.1) \quad |h(t)f(\varphi(x)) + h(1 - t)f(\varphi(y)) - f(t\varphi(x) + (1 - t)\varphi(y))| \\ \leq h(t)g(\varphi(x)) + h(1 - t)g(\varphi(y)) - g(t\varphi(x) + (1 - t)\varphi(y))$$

for all  $x, y \in I$  and  $t \in (0, 1)$ .

In particular, if  $f$  satisfies (2.1) with  $h(t) = t$ ,  $h(t) = t^s$  ( $s \in (0, 1)$ ),  $h(t) = \frac{1}{t}$  and  $h(t) = 1$ , then  $f$  is said to be  $(g, \varphi)$ -convex-dominated,  $(g, \varphi_s)$ -convex-dominated,  $(g, \varphi_{Q(I)})$ -convex-dominated and  $(g, \varphi_{P(I)})$ -convex-dominated functions, respectively.

The next simple characterisation of  $(g, \varphi_h)$ -convex dominated functions holds.

**Lemma 1.** *Let  $h : (0, 1) \rightarrow (0, \infty)$  be a given function,  $g : I \rightarrow [0, \infty)$  be a given  $\varphi_h$ -convex function and  $f : I \rightarrow [0, \infty)$  be a real function. The following statements are equivalent:*

- (1)  $f$  is  $(g, \varphi_h)$ -convex dominated on  $I$ .
- (2) The mappings  $g - f$  and  $g + f$  are  $\varphi_h$ -convex on  $I$ .
- (3) There exist two  $\varphi_h$ -convex mappings  $l, k$  defined on  $I$  such that

$$f = \frac{1}{2}(l - k) \quad \text{and} \quad g = \frac{1}{2}(l + k) .$$

*Proof.*  $1 \iff 2$  The condition (2.1) is equivalent to

$$\begin{aligned} & g(t\varphi(x) + (1-t)\varphi(y)) - h(t)g(\varphi(x)) - h(1-t)g(\varphi(y)) \\ & \leq h(t)f(\varphi(x)) + h(1-t)f(\varphi(y)) - f(t\varphi(x) + (1-t)\varphi(y)) \\ & \leq h(t)g(\varphi(x)) + h(1-t)g(\varphi(y)) - g(t\varphi(x) + (1-t)\varphi(y)) \end{aligned}$$

for all  $x, y \in I$  and  $t \in [0, 1]$ . The two inequalities may be rearranged as

$$\begin{aligned} & (g + f)(t\varphi(x) + (1-t)\varphi(y)) \\ & \leq h(t)(g + f)(\varphi(x)) + h(1-t)(g + f)(\varphi(y)) \end{aligned}$$

and

$$\begin{aligned} & (g - f)(t\varphi(x) + (1-t)\varphi(y)) \\ & \leq h(t)(g - f)(\varphi(x)) + h(1-t)(g - f)(\varphi(y)) \end{aligned}$$

which are equivalent to the  $\varphi_h$ -convexity of  $g + f$  and  $g - f$ , respectively.

$2 \iff 3$  Let we define the mappings  $f, g$  as  $f = \frac{1}{2}(l - k)$  and  $g = \frac{1}{2}(l + k)$ . Then if we sum and subtract  $f$  and  $g$ , respectively, we have  $g + f = l$  and  $g - f = k$ . By the condition 2 in Lemma 1, the mappings  $g - f$  and  $g + f$  are  $\varphi_h$ -convex on  $I$ , so  $l, k$  are  $\varphi_h$ -convex mappings on  $I$  too.  $\square$

**Theorem 1.** *Let  $h : (0, 1) \rightarrow (0, \infty)$  be a given function,  $g : I \rightarrow [0, \infty)$  be a given  $\varphi_h$ -convex function. If  $f : I \rightarrow [0, \infty)$  is Lebesgue integrable and  $(g, \varphi_h)$ -convex dominated on  $I$  for linear continuous function  $\varphi : [a, b] \rightarrow [a, b]$ , then the following inequalities hold:*

$$\begin{aligned} (2.2) \quad & \left| \frac{1}{\varphi(b) - \varphi(a)} \int_{\varphi(a)}^{\varphi(b)} f(x) dx - \frac{1}{2h(\frac{1}{2})} f\left(\frac{\varphi(a) + \varphi(b)}{2}\right) \right| \\ & \leq \frac{1}{\varphi(b) - \varphi(a)} \int_{\varphi(a)}^{\varphi(b)} g(x) dx - \frac{1}{2h(\frac{1}{2})} g\left(\frac{\varphi(a) + \varphi(b)}{2}\right) \end{aligned}$$

and

$$(2.3) \quad \left| [f(\varphi(a)) + f(\varphi(b))] \int_0^1 h(t) dt - \frac{1}{\varphi(b) - \varphi(a)} \int_{\varphi(a)}^{\varphi(b)} f(x) dx \right|$$

$$\leq [g(\varphi(a)) + g(\varphi(b))] \int_0^1 h(t) dt - \frac{1}{\varphi(b) - \varphi(a)} \int_{\varphi(a)}^{\varphi(b)} g(x) dx$$

for all  $x, y \in I$  and  $t \in [0, 1]$ .

*Proof.* By the Definition 5 with  $t = \frac{1}{2}$ ,  $x = \lambda a + (1 - \lambda)b$ ,  $y = (1 - \lambda)a + \lambda b$  and  $\lambda \in [0, 1]$ , as the mapping  $f$  is  $(g, \varphi_h)$ -convex dominated function, we have that

$$\left| h\left(\frac{1}{2}\right) [f(\varphi(\lambda a + (1 - \lambda)b)) + f(\varphi((1 - \lambda)a + \lambda b))] - f\left(\frac{\varphi(\lambda a + (1 - \lambda)b) + \varphi((1 - \lambda)a + \lambda b)}{2}\right) \right|$$

$$\leq$$

$$h\left(\frac{1}{2}\right) [g(\varphi(\lambda a + (1 - \lambda)b)) + g(\varphi((1 - \lambda)a + \lambda b))] - g\left(\frac{\varphi(\lambda a + (1 - \lambda)b) + \varphi((1 - \lambda)a + \lambda b)}{2}\right).$$

Then using the linearity of  $\varphi$ -function, we have

$$\left| h\left(\frac{1}{2}\right) [f(\lambda\varphi(a) + (1 - \lambda)\varphi(b)) + f((1 - \lambda)\varphi(a) + \lambda\varphi(b))] - f\left(\frac{\varphi(a) + \varphi(b)}{2}\right) \right|$$

$$\leq$$

$$h\left(\frac{1}{2}\right) [g(\lambda\varphi(a) + (1 - \lambda)\varphi(b)) + g((1 - \lambda)\varphi(a) + \lambda\varphi(b))] - g\left(\frac{\varphi(a) + \varphi(b)}{2}\right).$$

If we integrate the above inequality with respect to  $\lambda$  over  $[0, 1]$ , the inequality in (2.2) is proved.

To prove the inequality in (2.3), firstly we use the Definition 5 for  $x = a$  and  $y = b$ , we have

$$|h(t)f(\varphi(a)) + h(1 - t)f(\varphi(b)) - f(t\varphi(a) + (1 - t)\varphi(b))|$$

$$\leq$$

$$h(t)g(\varphi(a)) + h(1 - t)g(\varphi(b)) - g(t\varphi(a) + (1 - t)\varphi(b)).$$

Then, we integrate the above inequality with respect to  $t$  over  $[0, 1]$ , we get

$$\left| f(\varphi(a)) \int_0^1 h(t) dt + f(\varphi(b)) \int_0^1 h(1 - t) dt - \int_0^1 f(t\varphi(a) + (1 - t)\varphi(b)) dt \right|$$

$$\leq g(\varphi(a)) \int_0^1 h(t) dt + g(\varphi(b)) \int_0^1 h(1 - t) dt - \int_0^1 g(t\varphi(a) + (1 - t)\varphi(b)) dt.$$

If we substitute  $x = t\varphi(a) + (1 - t)\varphi(b)$  and use the fact that  $\int_0^1 h(t) dt = \int_0^1 h(1 - t) dt$ , we get

$$\left| [f(\varphi(a)) + f(\varphi(b))] \int_0^1 h(t) dt - \frac{1}{\varphi(b) - \varphi(a)} \int_{\varphi(a)}^{\varphi(b)} f(x) dx \right|$$

$$\leq [g(\varphi(a)) + g(\varphi(b))] \int_0^1 h(t) dt - \frac{1}{\varphi(b) - \varphi(a)} \int_{\varphi(a)}^{\varphi(b)} g(x) dx.$$

So, the proof is completed.  $\square$

**Corollary 1.** *Under the assumptions of Theorem 1 with  $h(t) = t$ ,  $t \in (0, 1)$ , we have*

$$(2.4) \quad \left| \frac{1}{\varphi(b) - \varphi(a)} \int_{\varphi(a)}^{\varphi(b)} f(x) dx - f\left(\frac{\varphi(a) + \varphi(b)}{2}\right) \right| \\ \leq \frac{1}{\varphi(b) - \varphi(a)} \int_{\varphi(a)}^{\varphi(b)} g(x) dx - g\left(\frac{\varphi(a) + \varphi(b)}{2}\right)$$

and

$$(2.5) \quad \left| \frac{f(\varphi(a)) + f(\varphi(b))}{2} - \frac{1}{\varphi(b) - \varphi(a)} \int_{\varphi(a)}^{\varphi(b)} f(x) dx \right| \\ \leq \frac{g(\varphi(a)) + g(\varphi(b))}{2} - \frac{1}{\varphi(b) - \varphi(a)} \int_{\varphi(a)}^{\varphi(b)} g(x) dx.$$

**Remark 1.** *If function  $\varphi$  is the identity in (2.4) and (2.5), then they reduce to Hermite-Hadamard type inequalities for convex dominated functions proved by Dragomir, Pearce and Pečarić in [2].*

**Corollary 2.** *Under the assumptions of Theorem 1 with  $h(t) = t^s$ ,  $t, s \in (0, 1)$ , we have*

$$(2.6) \quad \left| \frac{1}{\varphi(b) - \varphi(a)} \int_{\varphi(a)}^{\varphi(b)} f(x) dx - 2^{s-1} f\left(\frac{\varphi(a) + \varphi(b)}{2}\right) \right| \\ \leq \frac{1}{\varphi(b) - \varphi(a)} \int_{\varphi(a)}^{\varphi(b)} g(x) dx - 2^{s-1} g\left(\frac{\varphi(a) + \varphi(b)}{2}\right)$$

and

$$(2.7) \quad \left| \frac{f(\varphi(a)) + f(\varphi(b))}{s+1} - \frac{1}{\varphi(b) - \varphi(a)} \int_{\varphi(a)}^{\varphi(b)} f(x) dx \right| \\ \leq \frac{g(\varphi(a)) + g(\varphi(b))}{s+1} - \frac{1}{\varphi(b) - \varphi(a)} \int_{\varphi(a)}^{\varphi(b)} g(x) dx.$$

**Remark 2.** *If function  $\varphi$  is the identity in (2.6) and (2.7), then they reduce to Hermite-Hadamard type inequalities for  $(g, s)$ -convex dominated functions proved by Kavurmacı, Özdemir and Sarıkaya in [4].*

**Corollary 3.** *Under the assumptions of Theorem 1 with  $h(t) = \frac{1}{t}$ ,  $t \in (0, 1)$ , we have*

$$(2.8) \quad \left| \frac{4}{\varphi(b) - \varphi(a)} \int_{\varphi(a)}^{\varphi(b)} f(x) dx - f\left(\frac{\varphi(a) + \varphi(b)}{2}\right) \right| \\ \leq \frac{4}{\varphi(b) - \varphi(a)} \int_{\varphi(a)}^{\varphi(b)} g(x) dx - g\left(\frac{\varphi(a) + \varphi(b)}{2}\right).$$

**Remark 3.** If function  $\varphi$  is the identity in (2.8), then it reduces to Hermite-Hadamard type inequality for  $(g, Q(I))$ -convex dominated functions proved by Özdemir, Tunç and Kavurmacı in [6].

**Corollary 4.** Under the assumptions of Theorem 1 with  $h(t) = 1$ ,  $t \in (0, 1)$ , we have

$$(2.9) \quad \left| \frac{2}{\varphi(b) - \varphi(a)} \int_{\varphi(a)}^{\varphi(b)} f(x) dx - f\left(\frac{\varphi(a) + \varphi(b)}{2}\right) \right| \leq \frac{1}{\varphi(b) - \varphi(a)} \int_{\varphi(a)}^{\varphi(b)} g(x) dx - g\left(\frac{\varphi(a) + \varphi(b)}{2}\right)$$

and

$$(2.10) \quad \left| [f(\varphi(a)) + f(\varphi(b))] - \frac{1}{\varphi(b) - \varphi(a)} \int_{\varphi(a)}^{\varphi(b)} f(x) dx \right| \leq [g(\varphi(a)) + g(\varphi(b))] - \frac{1}{\varphi(b) - \varphi(a)} \int_{\varphi(a)}^{\varphi(b)} g(x) dx.$$

**Remark 4.** If function  $\varphi$  is the identity in (2.9) and (2.10), then they reduce to Hermite-Hadamard type inequalities for  $(g, P(I))$ -convex dominated functions proved by Özdemir, Tunç and Kavurmacı in [6].

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